Lecture 5. Poisson's and Laplace's equations. The curl of $\underline{E}$.

Last time:

① Gauss's law in differential form:

$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$ → Give me $\underline{E}$ & I can calculate ρ. Easy way to go from $\underline{E}$ to the charge distribution that created it.
→ Eq #1 from Maxwell's equations

② Electric potential

$\varphi(\underline{r}) = -\int_{\infty}^{r} \underline{E}\cdot d\underline{s}$ work done to move a unit charge from ∞ to point $\underline{r}$

$\underline{E} = -\nabla\varphi$ It's a scalar, much easier to work with than $\underline{E}$.

Today: Deep dive on applications of vector calculus to electrostatics (i.e the math needed to describe the physics)

Electrostatics → Find $\underline{E}$ of a given stationary charge distribution
charges DON'T move

How do we do this?

$\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\underline{r})}{r^2}\, dV$ → ugly integrals
→ sometimes can be solved by applying Gauss's law ONLY when the problem has symmetries.

Strategy we learnt last class:

$\varphi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\rho(\underline{r})\, dV$ we defined a scalar potential

advantage: scalar function

disadvantages: integrals are still ugly
what happens if we don't know ρ in advance? conductors (next week) charge is free to move

Another helpful strategy: Recast the problem in differential form

We start with Gauss's law in differential form:

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$$

and use our definition of potential:

$$\underline{E} = -\nabla\varphi$$

Then:

$$\nabla \cdot (-\nabla\varphi) = \frac{1}{\varepsilon_0}\rho$$

$$\Rightarrow \boxed{\nabla^2\varphi = \frac{-1}{\varepsilon_0}\rho}$$ Poisson's equation

In regions where there is no charge this reduces to:

$$\boxed{\nabla^2\varphi = 0}$$ Laplace's equation

Let's take a step back. The Laplacian operator

If we go back to the definition of the "del" operator (lecture 1)

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\nabla \cdot \nabla f = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f \equiv \nabla^2 f$$

$$\boxed{\nabla^2 f \equiv \nabla \cdot \nabla f}$$

Laplacian operator !! "The divergence of a gradient"

- second derivative
- gives the curvature of a function

beware of curvilinear coordinates where you might have some extra terms in the derivatives. See Purcell apendix F (F.1)

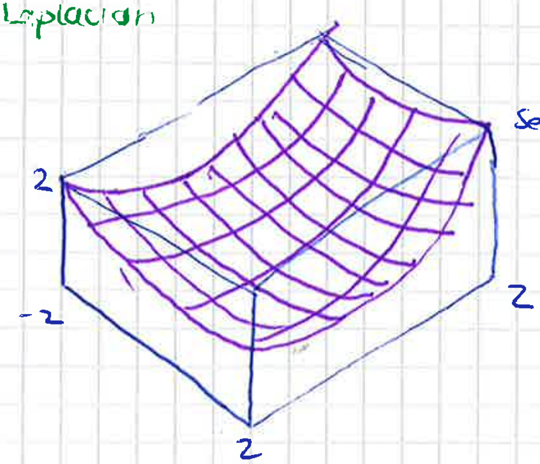
Geometrical interpretation of the Laplacian

Consider $\varphi(x,y) = \frac{a(x^2+y^2)}{4}$

$$\nabla^2 \varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$$= \frac{a}{4}(2+2) = a$$

curvature of a function



What does this mean in terms of Laplace's equation?

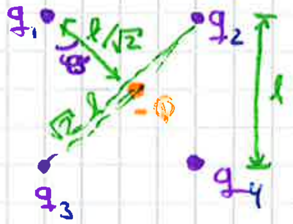
$\nabla^2 \varphi = 0 \Rightarrow$ If φ satisfies Laplace's eq, then its curvature must be zero everywhere in the region

$\Rightarrow$ the potential has no local maxima or minima in that region.

What does this mean physically?

Earnshaw's theorem:
It's impossible to hold a charge in stable equilibrium with electrostatic fields (i.e. the potential has no minima)

Let's think about the problem from lecture 2: Find Q such that the ΣF @ the corners is zero



Is this system in stable equilibrium? Let's compute the energy

$$V = \frac{k}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_i q_j}{r_{ij}} \qquad \text{where } k = \frac{1}{4\pi\varepsilon_0}$$

$$= k\left[\frac{4(-Q)q}{\ell/\sqrt{2}} + \frac{4q^2}{\ell} + \frac{2q^2}{\sqrt{2}\,\ell}\right] = \frac{4\sqrt{2}\,q\,k}{\ell}\left(-Q + \frac{q}{\sqrt{2}} + \frac{q}{4}\right) = 0$$

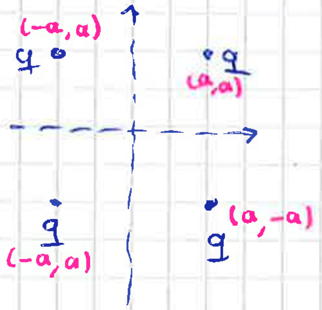
$$\Rightarrow Q = q\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right)$$ which is what we found before.

$\therefore V = 0$

we found that
$Q = q\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right) = 0.957q$
for $\Sigma F_i = 0$ on the corner

$$|F| = k\left[2\frac{q^2}{\ell^2}\cos\theta + \frac{q^2}{(\sqrt{2}\,\ell)^2} - \frac{qQ}{(\ell/\sqrt{2})^2}\right] = 0$$

But is this system in stable equilibrium?



$$U(x,y) = K_1 \left\{ \frac{1}{[(a-x)^2+(a-y)^2]^{1/2}} + \frac{1}{[(a-x)^2+(-a-y)^2]^{1/2}} + \frac{1}{[(-a-x)^2+(a-y)^2]^{1/2}} + \frac{1}{[(-a-x)^2+(-a-y)^2]^{1/2}} \right\}$$

where $K_1 = \frac{-Qq}{4\pi\varepsilon_0}$

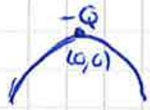
distances from q's to the point (x, y) are given by:

$$\left[(\pm a - x)^2 + (\pm a - y)^2\right]^{1/2}$$

then re-write this so I can Taylor expand it ...

$$U(x,y) \simeq \frac{-Qq}{4\pi\varepsilon_0 a}\left(2\sqrt{2} + \frac{x^2+y^2}{2\sqrt{2}a^2}\right)$$

the energy decreases with x & y, so the equilibrium is unstable for any direction of motion of −Q in the xy plane.



instead of

Did we need to go through this?

NO, just use Earnshaw's theorem.

More on Laplace's equation

$\nabla^2\varphi = 0$ → ubiquitous equation
- → 2D steady-state heat conduction
- → static deflection of a membrane
- → irrotational flows
- → Electrostatic potential

solutions to this eq. are called harmonic functions

(we know that $1/r$ is a solution!)

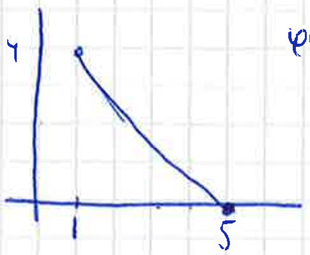
In 1D:

$\frac{d^2\varphi}{dx^2} = 0$ has a general solution $\varphi(x) = mx + b$

ordinary differential eq.

where m & b are const and determined by the boundary conditions of the problem

example of BC: $\varphi(1) = 4$ and $\varphi(5) = 0$ $\Rightarrow \varphi(x) = -x + 5$



$\varphi(x)$ is the average of $\varphi(x+a)$ and $\varphi(x-a)$ $\forall a$

$$\varphi(x) = \frac{1}{2}\left[\varphi(x+a) + \varphi(x-a)\right]$$

assign $\varphi(x)$ the average of the values to the left & right of it.

In 2D:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

partial differential eq.

It's not possible to write a general solution in closed form (i.e. doesn't have a simple analytical solution)

properties of the 1D eq hold:
- Value of φ at (x, y) is equal to the average value of φ around this point.

$$\varphi(x,y) = \frac{1}{2\pi R}\oint \varphi\, dl = \frac{1}{2\pi R}\oint \varphi R\, d\Phi$$

↳ path integral along a circle of arbitrary R centered at (x, y)

- φ has no local max or min, all extrema occur at the boundaries

↳ "no hills or valleys" just the smoothest possible surface.

- Minimizes the surface area spanning a given boundary

Demo 155 soap films

In 3D:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$$

No simple analytical solution or intuition for 3D

Properties of the solution in 1D & 2D still hold:

- Value of φ at (x,y,z) is equal to the average value of φ around this point

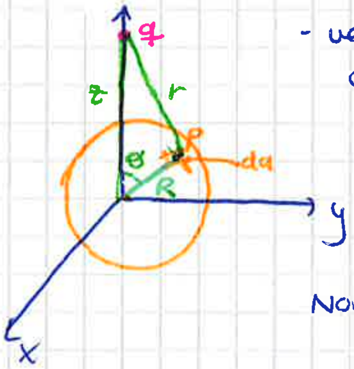
$$\varphi(x,y,z)=\frac{1}{4\pi R^2}\oint_{\text{sphere}}\varphi\, da=\frac{1}{4\pi R^2}\oint_{\text{sphere}}\varphi R^2\sin\theta\, d\theta\, d\Phi$$

↑ surface integral is across the surface of a sphere of arbitrary radius centered at (x,y,z) with radius R.

- φ has no local min or max. Extrema occur at the boundary.

We can prove the first property considering the electrostatic potential.

Let's calculate the electrostatic potential over a spherical surface of radius R due to a single charge located outside the sphere:



- we define our coord system such that q lies on the z axis and center the sphere at the origin.

- the potential generated by q felt at a pt. P on the surface:

$$\varphi_P=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$ where r is the distance between q & P

Now using trig we can express r in terms of z, R and θ

$$r^2=z^2+R^2-2zR\cos\theta$$

⇒ potential at point P:

$$\varphi_P=\frac{1}{4\pi\varepsilon_0}\frac{q}{[z^2+R^2-2zR\cos\theta]^{1/2}}$$

and we can obtain the average potential on the surface of the sphere by integrating V_P across the surface of the sphere:

$$\varphi_{AV}=\frac{1}{4\pi R^2}\int_\theta \varphi_P R^2\sin\theta\, d\theta\, d\Phi=\frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int[z^2+R^2-2zR\cos\theta]^{-1/2}R^2\sin\theta\, d\theta\, d\Phi$$

$$=\frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}[z^2+R^2-2zR\cos\theta]^{1/2}\Big|_0^\pi$$

$$=\frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}[(z+R)-(z-R)]=\frac{1}{4\pi\varepsilon_0}\frac{q}{z}$$ ← this is the same as the potential due to q at the center of the sphere.

If we apply the principle of superposition we can see that the average φ generated by a collection of pt charges is equal to the net potential they produce at the center of the sphere.

Example: Find the general solution to Laplace's equation in spherical coordinates for the case that φ only depends on r

In spherical coord Laplace's eq becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\cancelto{0}{\frac{\partial\varphi}{\partial\theta}}\right)+\frac{1}{r^2\sin^2\theta}\cancelto{0}{\frac{\partial^2\varphi}{\partial\Phi^2}}=0$$

$\varphi=\varphi(r)$

so Laplace's eq. reduces to:

$$\frac{1}{r^2}\underbrace{\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right)}_{}=0$$

If this derivative is zero $\Rightarrow$ $r^2\frac{\partial \varphi}{\partial r}$ is a constant

6

$\Rightarrow r^2\frac{\partial \varphi}{\partial r} = K$ where K is a constant

$\Rightarrow \frac{\partial \varphi}{\partial r} = \frac{K}{r^2}$ $\Rightarrow d\varphi = \int \frac{K}{r^2}\, dr$

$\Rightarrow \varphi = -\frac{K}{r} + b$ where b is a const

if we follow the convention where $\varphi(\infty) = 0$

$$\Rightarrow \varphi(r) = -\frac{K}{r}$$

Boundary conditions & the uniqueness theorem

$\nabla^2\varphi = 0$ we need appropriate boundary conditions to determine φ

First uniqueness theorem: The solution to Laplace's equation in some volume V is UNIQUELY determined if φ is specified on the boundary surfaces.

φ is specified at this surface

let's assume that there $\exists$ two solutions to Laplace's eq. that fulfill the specified value at the surface.

$$\nabla^2\varphi_1 = 0 \quad , \quad \nabla^2\varphi_2 = 0$$

let $\varphi_3 = \varphi_1 - \varphi_2$

because φ_1 & φ_2 obey Laplace's eq

$$\Rightarrow \nabla^2\varphi_3 = \nabla^2\varphi_1 - \nabla^2\varphi_2 = 0$$

since φ_1 & φ_2 are equal at the boundary by construction

$\Rightarrow \varphi_3 = 0$ at the boundary

Because φ_3 is a solution of Laplace's eq $\Rightarrow$ its min & max are located at the boundary $\Rightarrow$ max & min of φ_3 are both zero $\Rightarrow \varphi_3$ is zero everywhere

$$\Rightarrow \varphi_1 = \varphi_2$$

Why should I care about this?

It doesn't matter how you arrive at your solution if φ:

- φ satisfies Laplace's eq
- has the correct value at the boundaries $\Rightarrow$ it's the solution

such that the potential at the boundary location has the desired value.

Usefulness of this statement will become evident when using the method of images.

we can replace a problem containing boundaries by one without a boundary but w/ additional charge added

First uniqueness theorem also applies to Poisson's equation! Throw in some charge + consider φ_1 & φ_2 both solutions to Poisson's eq:

$$\nabla^2\varphi_1 = -\frac{1}{\varepsilon_0}\rho \quad , \quad \nabla^2\varphi_2 = -\frac{1}{\varepsilon_0}\rho$$

Let $\varphi_3 = \varphi_1 - \varphi_2 \Rightarrow \nabla^2\varphi_3 = \nabla^2\varphi_1 - \nabla^2\varphi_2 = -\frac{1}{\varepsilon_0}\rho + \frac{1}{\varepsilon_0}\rho = 0$

$\Rightarrow \varphi_3$ satisfies Laplace's equation

I can make the same argument about the min & max of φ_3 located at the boundary & since φ_3 at the boundary is zero $\Rightarrow \varphi_3$ is zero everywhere $\Rightarrow \varphi_1 = \varphi_2$

Corollary: The potential in a volume V is uniquely determined if:
- the charge density through the region and
- the value of φ on all boundaries are specified

Stokes theorem

Analog of the divergence theorem (From last lecture: $\int_V (\nabla\cdot \underline{v})dV = \oint_S \underline{v}\cdot d\underline{a}$)

$$\boxed{\int_S (\nabla\times\underline{v})\cdot d\underline{a} = \oint_P \underline{v}\cdot d\underline{l}}$$

Relates a surface integral of a derivative (in this case the curl) with the value of the function at the boundary.

Geometrical interpretation:

What is the curl? → Measures the twist of vectors $\underline{v}$.

region w/ ↑ curl
whirlpool

we can think of the $\int$ of the curl over a surface as the total amount of "swirl"

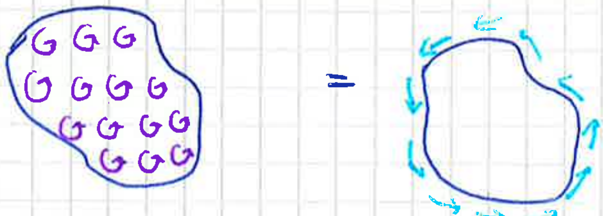
More precisely we know $\int_S f\cdot d\underline{a} = \Phi_f$ so in this case we have the "flux of the curl" over that surface.

What about the second term?

$\oint_P \underline{v}\cdot d\underline{l}$? This is called the circulation

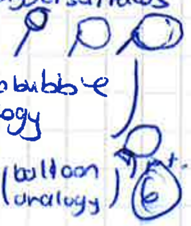
"Going around the edge & finding how much the flow is following the boundary.

We could picture the statement of Stokes theorem as:



$$\int_S (\nabla\times\underline{v})\cdot d\underline{a} = \oint_P \underline{v}\cdot d\underline{l}$$

same boundary, bigger surfaces



Corollary 1: $\int(\nabla\times\underline{v})\cdot d\underline{a}$ depends only on the boundary line, not particular surface used (soap bubble analogy)

Corollary 2: $\oint(\nabla\times\underline{v})\cdot d\underline{a} = 0$ for a closed surface, since the boundary line shrinks to a point (balloon analogy)

Application of stokes theorem

Because $\underline{F}_c$ is conservative we have that

$\oint \underline{F} \cdot d\underline{s} = 0$ (Lecture 3) $\Leftrightarrow \oint \underline{E} \cdot d\underline{s} = 0$

$\Rightarrow \int_S \nabla \times \underline{E} \cdot d\underline{a} = 0 \quad \Rightarrow \boxed{\nabla \times \underline{E} = 0}$ the curl of the electrostatic field is zero.

using stoke's theorem

are you surprised by this?

- we used this implicitly, in our definition of potential
  $\oint \underline{E} \cdot d\underline{s} = 0 \Rightarrow \int \underline{E} \cdot d\underline{s}$ is path independent
  so we can define a potential φ in an unambiguous manner $\left(\Delta\varphi \equiv -\int_{P_1}^{P_2} \underline{E} \cdot d\underline{s} \right)$
  recall

- Also in general we have that:
  $\nabla \times (\nabla f) = 0$
  the curl of a gradient is zero

Distinguishing the Physics from the math

What have we learned so far? (Griffiths fig. 2.35)